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IT TAKES t* TO TANGO:
TRADING COALITIONS IN THE EDGEWORTH PROCESS

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Abstract

In the Edgeworth non-tâtonnement process, trade occurs if there exists some coalition of agents able to make a Pareto-improving trade among themselves at current prices. It is known that the size of such coalitions is bounded by the number of commodities and that, provided all agents always have strictly positive endowments, bilateral trade suffices. These results are generalized: Let there be h agents, k of whom have strictly positive endowments. Let there be m commodities, n of which are held by all agents. Then the Edgeworth-process requires coalitions with at most $t^* = \text{Min}\ \{2,\ \text{Max}\ (h - k,\ m - n)\}$ members. This is a least upper bound.

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1. Introduction

The basic assumption of the Edgeworth non-tâtonnement process is that trade takes place if and only if there exists a coalition of agents able to make a Pareto-improving trade among themselves at current, disequilibrium prices. Among other objections to this assumption is the possibility that it may require a very large number of agents to find each other (Fisher, 1976, p. 19, 1983, pp. 29-31). In reply to this, David Schmeidler has observed (in a private communication) that such trading coalitions need never involve more members than the number of commodities, while Paul Madden has shown that, if all agents always have strictly positive endowments of all commodities, then such coalitions need never have more than two members. (Both results can be found in Madden, 1978).

These are not very reassuring answers to the problem at hand, however, particularly if one thinks of extending the Edgeworth process to relatively realistic settings.. If consumption takes place at different times, then the same commodity at different dates will be treated as different commodities. This can easily make the number of commodities much greater than the number of agents in the economy. As for Madden's bilateral trade result, it requires strictly positive endowments of all commodities for all agents, and this is far too strong a requirement in the context of disequilibrium trade.

It is therefore of some interest to see the extent to which the two existing results can be generalized. It turns out to be possible to accomplish this with a very elementary proof, and,

while the results still do not suggest that the Edgeworth-process assumption is free of coalition-formation problems, they may have some intrinsic interest.

I show the following under very general assumptions. Let there be h households of whom k have strictly positive endowments of all commodities. Let there be m commodities of which n are held in positive amounts by all households. Then the maximum number of agents who must participate in an Edgeworth-process trade is $t^* = \text{Max}\ \{2,\ \text{Min}\ (h - k,\ m - n)\}$. Further, this is a least upper bound: There exist examples requiring participation by t^* traders.

It is obvious that these results generalize and strengthen those of Schmeidler and Madden. Not surprisingly, they are also quite similar to results on the closely related question of when "t-wise optimality" -- the non-existence of Pareto-improving trades involving no more than t traders for some arbitrary t -- is equivalent to full Pareto optimality. (See Feldman, 1973, Graham, Jennergen, Peterson, and Weintraub, 1976, Madden, 1975, Rader, 1968, 1976, and, especially, Goldman and Starr, 1982.) The difference is that, in the Edgeworth process, trading is restricted to take place at given prices, so the theorems of the t-wise optimality literature cannot be used directly.

2. Preliminaries

There are h households and m commodities. Each household has a weakly monotonic, differentiable, quasi-concave utility function.[1]

Under these assumptions, an Edgeworth-process trade can be

thought of as a circle of agents and commodities. That is, every such trade involves a set of agents, which we may as well take to be {1, . . ., t}, and a set of commodities, which we may as well take to be also {1, . . ., t}, such that, for $1 \leq i < t$, household i sells commodity i to household i+1, while household t sells commodity t to household 1. The question at issue is that of when the size of such circles can be reduced.

The following fairly obvious fact will be central to the proofs below.

Lemma 1. Consider any household, H, and any triplet of commodities, a, b, c, with H's holdings of a and b both positive. Suppose that, at current prices, H could increase utility by selling a and buying c. Then, at the same prices, H would also find one of the following trades to be utility-increasing: (1) selling b and buying c or (2) selling a and buying b.

Proof. Denote H's utility function by U(.) and derivatives by subscripts in the obvious way. Let the prices of the three goods be p_a, p_b, and p_c, respectively. Then $U_a/U_c < p_a/p_c$, since H could increase utility by selling a and buying c. Evidently, either $U_b/U_c < p_b/p_c$, in which case H would find selling b and buying c to be utility increasing, or else $U_a/U_b < p_a/p_b$, in which case H would find selling a and buying b to be utility increasing.

3. Results

I begin with two parallel lemmata. (Cf. Goldman and Starr, 1982, pp. 597-598.)

Lemma 2. Suppose that there is an Edgeworth-process trade involving a household with strictly positive stocks of all the

commodities involved in the trade. Then there is an Edgeworth-process trade that involves no more than two households.

Proof. Without loss of generality, renumber households and commodities so that the assumed Edgeworth-process trade involves households and commodities $\{1, \ldots, t\}$, as described above. Suppose $t > 2$ and that household 1 has a positive endowment of the first t commodities. Household 1 finds it utility-improving to sell commodity 1 (to household 2) and buy commodity t (from household t).

Consider commodity t-1 which is being bought by household t and sold by household t-1. If household 1 would find it utility increasing to sell commodity t-1 and buy commodity t, then a bilateral Edgeworth-process trade is possible between households 1 and t. Suppose, on the other hand, that this is not the case. Then, by Lemma 1, household 1 would find it utility improving to sell commodity 1 and buy commodity t-1. In this case, however, there is an Edgeworth-process trade that involves only households $\{1, \ldots, t-1\}$ and the identically-numbered commodities. Repetition of this argument proves the lemma.

Lemma 3. Suppose that there is an Edgeworth-process trade involving a commodity that is held in strictly positive amounts by all the households involved in the trade. Then there is an Edgeworth-process trade that involves no more than two households.

Proof. As before, let the households and commodities involved in the Edgeworth-process trade be numbered $\{1, \ldots, t\}$. Assume that $t > 2$ and that it is commodity 1 that is held in

positive amounts by the first t households. Household t finds it utility increasing to sell commodity t to household 1 and buy commodity t-1 from household t-1.

Household t has a positive stock of commodity 1. If it would find it utility increasing to buy commodity 1 and sell commodity t, then a bilateral Edgeworth-process trade is possible between households 1 and t. Suppose, on the other hand, that this is not the case. Then, by Lemma 1, household t would find it utility-increasing to sell commodity 1 and buy commodity t-1. In this case, however, there is an Edgeworth-process trade that involves only households {2, . . ., t} and the identically-numbered commodities. Repetition of this argument proves the lemma.

It is now easy to prove the main result:

Theorem 1. Suppose that k of the households have positive stocks of all commodities and that n of the commodities are held in positive amounts by all households. Define $t^* = \text{Max}\ \{2, \text{Min}\ (h - k, m - n)\}$.

(A) If there exists and Edgeworth-process trade, then there exists one that involves no more than t* households.

(B) t* is a least upper bound to the number of participants that can be required in an Edgeworth-process trade, that is, there exist cases in which t* participants are necessary.

Proof. (A) Any Edgeworth-process trade that involves more than t* households must involve either a household that has positive stocks of all commodities or a commodity that is held in positive amounts by all households. Lemmas 2 and 3 show that

there must then be an Edgeworth-process trade involving only two households.

(B) This part of the theorem can be proved by constructing examples in which t^* participants are required.

If $t^* = 2$, the result is trivial. So suppose that $t^* = \text{Min}(h - k, m - n)$. $t^* = h - k$. Let households $1, \ldots, t^*$ be such that household i has a positive stock of only commodity i. For $1 < i \leq t^*$, suppose that the only utility-increasing trade for household i at current prices would be to sell commodity i and buy commodity $i-1$. For household 1, suppose that the only utility-increasing trade at current prices would be to sell commodity 1 and buy commodity t^*. Finally, suppose that households $t^*+1, \ldots, h$ have no utility-increasing trade that can be made at current prices. Then the only Edgeworth-process trade is the obvious one involving the first t^* households and commodities and it cannot be reduced.

Theorem 1 plainly implies both Schmeidler's and Madden's results. Indeed, it permits us to strengthen the latter as:

Corollary 1. Let all but two of the households have a strictly positive amount of every commodity. Then, if there is an Edgeworth-process trade, there is one that is bilateral.

Corollary 2. Let all but two of the commodities be held in strictly positive amount by every household. Then, if there is an Edgeworth-process trade, there is one that is bilateral.

In closing, I note that the fact that Lemmas 2 and 3 speak only in terms of households and commodities involved in the assumed Edgeworth-process trade may mean that further results are

possible. I do not see how to phrase such results in an interesting way, however.

NOTE

1. The assumption of differentiability can almost certainly be weakened to the requirement that indifference surfaces have unique supporting hyperplanes (Madden, 1978, p. 281), but there seems little gain in complicating the exposition to do so. Apart from the method of proof used, one needs to rule out the following possibility. Suppose that household 1 regards apples and bananas as perfect complements while households 2 and 3 do not. In that circumstance, the three households may have a Pareto-improving trade in which 1 sells carrots to 2 for apples and to 3 for bananas. Such a trade can require three participants even though a particular household (1) participates in all transactions. This makes calculation of the minimum number of participants tedious at best, and, as the circumstance involved is quite special, it does not seem worth pursuing. (Note that if _all_ agents view a given subset of commodities as perfect complements using the same proportions, then, without loss of generality, that subset can be renamed as a composite commodity.)

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